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A NOTE ON TRANSLATIONS OF C INTO I

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A note on translations of C into I.

0. This note presents a stronger form of Glivenco's translation (prop. 14). The method used yields all the known translations of C into I, assuming Kolmogorov's translation as a starting point. The result is generalized (prop. 17), and the impossibility to obtain an "optimal" translation is shown.

1. Notation:

A, B, C, D, E denote formulas.

$\underline{A}$, $\underline{B}$ etc. - occurrences of formulas.

Λ - the symbol of absurdity.

S_A - the set of all occurrences of subformulas of A.

S_A^- - the set of all negative occurrences of subformulas of A.

S_A^+ - the set of all positive occurrences of subformulas of A.

S_A^{++} - the set of all strictly-positive occurrences of subformulas of A.

(cf. [Prawitz 65] for definitions).

I - the intuitionistic predicate calculus.

C - the classical predicate calculus.

If $\underline{B} \in S_A$, then $A(\frac{\underline{B}}{\underline{C}})$ is the formula which results from A by substituting $\underline{C}$ for $\underline{B}$. Similarly for $A(\frac{\beta}{\delta})$, where

$$\beta = \langle \underline{B}_1, \ldots, \underline{B}_K \rangle, \ \underline{B}_i \in S_A \ (1 \leq i \leq K); \ \delta = \langle \underline{D}_1, \ldots, \underline{D}_K \rangle.$$

Also: $\beta(\frac{\underline{B}_i}{\underline{C}}) =_{Df} \langle \underline{B}_1, \ldots, \underline{B}_{i-1}, \underline{C}, \underline{B}_{i+1}, \ldots, \underline{B}_K \rangle$,

and $\neg\neg\beta =_{Df} \langle \underline{\neg\neg B}_1, \ldots, \underline{\neg\neg B}_K \rangle$.

We call A a d-formula if either:

(i) A is a prime formula, or

(ii) the main logical symbol of A is $\vee$ or $\exists$.

2. Definitions:

On S_A define a partial order $\leq$ by:

$$\underline{B} \leq \underline{C} \equiv_{Df} \underline{C} \in S_{\underline{B}} .$$

$T_A =_{Df} \langle S_A, \leq \rangle$ is then a tree, which we call the formula-tree of A.

Clearly we can identify every point (i.e. - formula) of T_A with its main logical symbol.

$\beta = \{\underline{B}_1, \ldots, \underline{B}_K\} \subseteq T \subseteq S_A$ is a bar of T, if

(i) $\underline{B}_i$ and $\underline{B}_j$ are uncomparable under $\leq$ for $1 \leq i < j \leq K$.

(ii) every $\underline{C} \in T$ is comparable to some $\underline{B}_i$.

β is a clear bar if no $\underline{C} \in S_A$ s.t. $\underline{C} < \underline{B}_i$ (for some $1 \leq i \leq k$) is a d-formula.

The set of bars of $T \subseteq S_A$ is partially-ordered by

$$\beta_1 \leq \beta_2 \equiv_{Df} [\forall \underline{B}_i \in \beta_1 \ \forall \underline{B}_2 \in \beta_2 \ \neg[\underline{B}_2 < \underline{B}_1]].$$

Clearly every $T \subseteq S_A$ has a maximal clear bar in this ordering, the elements of which are either $\underline{\Lambda}$ or d-formulas.

β is free of x if every $\underline{B}_i$ $(1 \leq i \leq K)$ is free of x.

3. Lemma:

(a) Let $\underline{B} \in S_A^+$, and $B \to C \in I$, then $\vdash_I A \to A(\frac{B}{\underline{C}})$.

(b) Let $\underline{B} \in S_A^+$ have no free variable bounded in A by $\exists$, and C have no free variable bounded in A by $\forall$, then

$$B \to C \vdash_I A \to A(\frac{B}{\underline{C}}).$$

(c) Let $\underline{B} \in S_A^-$, and $C \to B \in I$, then $\vdash_I A \to A(\frac{B}{\underline{C}})$.

(d) Let $\underline{B} \in S_A^-$ and C be restricted as in (b), then

$$C \to B \vdash_I A \to A(\frac{B}{\underline{C}}).$$

<u>Proof</u>: (a) and (c):

Proceed by double-induction. The main induction is on the number of alternation between S_A^+ and S_A^- in the branch leading from $\underline{A}$ to $\underline{B}$ in S_A. To prove the basis use the following induction-steps in the natural-deduction system of [Prowitz 65] (Π denotes everywhere a deduction of I, by the induction-assumption).

(i)

$$\dfrac{\begin{matrix} D \,\&\, E \\ D \\ \Pi \\ D\left(\frac{\underline{B}}{\underline{C}}\right) \end{matrix} \qquad \dfrac{D\&E}{E}}{(D\&E)\left(\frac{\underline{B}}{\underline{C}}\right)}$$

(ii)

$$\dfrac{D \vee E \qquad \dfrac{\begin{matrix} (1) \\ D \\ \Pi \\ D\left(\frac{\underline{B}}{\underline{C}}\right) \end{matrix}}{(D\vee E)\left(\frac{\underline{B}}{\underline{C}}\right)} \qquad \begin{matrix} (2) \\ E \\ (D\vee E)\left(\frac{\underline{B}}{\underline{C}}\right) \end{matrix}}{(D\vee E)\left(\frac{\underline{B}}{\underline{C}}\right)}\ (1)(2)$$

(iii)

$$\begin{matrix} \forall x D x \\ Da \\ \Pi \\ Da\left(\frac{\underline{B}^x_a}{\underline{C}^x_a}\right) \\ (\forall x D x)\left(\frac{\underline{B}}{\underline{C}}\right) \end{matrix}$$

(iv)

$$\dfrac{\exists x D x \qquad \begin{matrix} (1) \\ Da \\ \Pi \\ Da\left(\frac{\underline{B}^x_a}{\underline{C}^x_a}\right) \\ (\exists x D x)\left(\frac{\underline{B}}{\underline{C}}\right) \end{matrix}}{(\exists x D x)\left(\frac{\underline{B}}{\underline{C}}\right)}\ (1)$$

(v)
$$
\begin{array}{c}
\dfrac{E \to D \qquad \overset{(1)}{E}}{D} \\
\Pi \\
D\left(\frac{B}{\underline{C}}\right) \\
(E \to D)\left(\frac{B}{\underline{C}}\right)
\end{array}
$$

For the main-induction inductive step we have to consider, in addition to the above, also the following case:

(vi) $D \in S_A^-$, and by the main-induction assumption $D(\frac{B}{\underline{C}}) \to D \in I$, hence

$$
\dfrac{\dfrac{\dfrac{\overset{\Sigma}{D(\frac{B}{\underline{C}}) \to D} \qquad \overset{(1)}{D(\frac{B}{\underline{C}})}}{D} \qquad D \to E}{E}}{(D \to E)(\frac{B}{\underline{C}})}\,(1) \quad .
$$

The main-induction inductive step for (c) is symmetric to (vi). This concludes the proof for (a) and (c).

The proof for (b) and (d) is similar. The restrictions on $\underline{B}$ and C result from the restrictions on the ∀I and ∃E-rules in cases (iii) and (iv).

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Remarks:

1. The lemma can be extended, using a trivial induction, to the replacement of sequences of occurrences-of-formulas.

2. Let $x_1 \dots x_K$ be the complete list of the free variables of B bounded in A by ∃, and of the free variables of C bounded in A by ∀. Then we clearly have:

(b') For $\underline{B} \in S_A^+$

$$\forall x_1 \dots \forall x_K\,(B \to C) \vdash_I A \to A\left(\frac{B}{\underline{C}}\right)$$

(without any additional restrictions on $\underline{B}$ and C. And analoguely - (d')).

The significance of the restrictions becomes apparent only when some property of $B \to C$ which $\forall x_1 \ldots x_K (B \to C)$ does not possess is used. For instance:

$$\vdash \neg\neg (B \to C) \quad \text{but} \quad \nvdash \neg\neg \forall x_1 \ldots x_K (B \to C).$$

4. Lemma:

The following are theorems of I:

(a) $\neg\neg (A \& B) \Longleftrightarrow \neg\neg A \,\&\, \neg\neg B$

(b) $\neg\neg (A \to B) \Longleftrightarrow (\neg\neg A \to \neg\neg B) \Longleftrightarrow (A \to \neg\neg B)$

(c) $(\neg\neg A \vee \neg\neg B) \to \neg\neg (A \vee B)$

(d) $\exists x \neg\neg A \to \neg\neg \exists x A$

(e) $\neg\neg \forall x A \to \forall x \neg\neg A$

(f) $\neg\neg \Lambda \to \Lambda$ equivalently: $\neg\neg\neg A \to \neg A$

(g) $A \to \neg\neg A$

(h) $\neg\neg (\neg\neg A \to A)$

Proof:

cf. [Kleene 52].

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5. Lemma (Kolmogorov 25)

Let $\overline{A}$ result from A by double-negating (inductively) every $\underline{B} \in S_A$.

then $\vdash_C A \Rightarrow \vdash_I \overline{A}$.

Proof:

Check (using lemma 4) for some formal systems generating I and C ([Prawitz 65] or [Kleene 52] for instance), that for every A which is an axiom of C, $\overline{A}$ is a theorem of I, and if $(\frac{A_i}{B})$ is a rule of inference for C, then $\bigwedge_i \overline{A}_i \to \overline{B}$ is a theorem of I.

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6. Lemma:

Let A^+ result from A by double-negating (inductively) every $B \in S_A^+$; then $\vdash_C A \Rightarrow \vdash_I A^+$.

Proof:

Delete inductively the double-negations of $\underline{B} \in S_{\overline{A}}$ in lemma 5; using 3(c) and 4(g).

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7. Proposition (Gödel 32)

Let A be s.t. every d-formula in S_A^+ is negated in A; then $\vdash_C A \Rightarrow \vdash_I A$.

Proof:

Assume $\vdash_C A$. By (6) $\vdash_I A^+$.

We eliminate now the double-negations added to S_A^+ to obtain A^+ by procedding inductively upwards in T_A. Let $\underline{B} \in S_A^+$. If B is a d-formula use the proposition's assumption, (4f) and (3a) to get

$\vdash_I A^+ \left(\frac{\neg\neg\underline{B}^+}{\underline{B}^+}\right)$.

If $B \equiv C\&D$, then by (4a)

$$\neg\neg\, B^+ \equiv \neg\neg\, (\neg\neg C^+ \& \neg\neg D^+) \to \neg\neg\neg\neg\, C^+ \,\&\, \neg\neg\neg\neg\, D^+$$

$$\to \neg\neg\, C^+ \,\&\, \neg\neg\, D^+ \quad \text{(by (4f))}.$$

Hence, again by (3a), $\vdash_I A^+ \left(\frac{\neg\neg B^+}{B^+}\right)$.

Similarly for B negational, implicational or universal, using (instead of (4a)) (4f), (4b) and (4e) respectively.

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8. Proposition (Glivenco 29, Minc-Orevkov 63):

Let A be sucht that no $\underline{B} \in S_A^+$ is a universal formula; then $\vdash_C A \Rightarrow \vdash_I \neg\neg\, A$.

Proof:

Symmetric to the proof of (7). We proceed inductively downwards in T_A, using (4a-d,f), to eliminate the double-negations in A^+.

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9. Corollary (Kreisel 58):

If A is a negation of a prenex formula, then $\vdash_C A \Rightarrow \vdash_I A$.

10. Proposition:

If for every $\underline{\forall xB} \in S_A^+$ we have

$$(*) \qquad \forall x \neg\neg B \rightarrow \neg\neg \forall xB,$$

then $\vdash_C A \Rightarrow \vdash_I \neg\neg A$.

Proof:

Like that of (8).

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Proposition (10) establishes incidentally that the intermediate logic MH, which arrises from I by the adjunction of (*) (understood as a scheme) is the minimal logic X s.t. $\vdash_C A \Rightarrow \vdash_X \neg\neg A$ for every first-order formula A.

11. Lemma:

If $\underline{\neg\neg C} \in S_B$ is free of x, then $\vdash_I \forall xB \rightarrow \neg\neg \forall xB\ (\frac{\neg\neg C}{\underline{C}})$.

Proof:

If $\underline{\neg\neg C} \in S_B^-$ the result follows immediately 3(c) and 4(g) (without the restriction on C).

If $\underline{\neg\neg C} \in S_B^+$, then, since C is free of x, there is by 3(b) a deduction Π, and by 4(h) a deduction Σ, s.t. the following is a proof (in the natural-deduction system of [Prawitz 65]):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)}{\forall xB}\qquad \overset{(2)}{\neg\neg C\to C}}{\forall xB(\frac{\neg\neg C}{\underline{C}})}\,\Pi \qquad \overset{(3)}{\neg\forall xB(\frac{\neg\neg C}{\underline{C}})}}{\Lambda}}{\neg(\neg\neg C\to C)}\,(2) \qquad \overset{\Sigma}{\neg\neg(\neg\neg C\to C)}}{\Lambda}}{\neg\neg\forall xB(\frac{\neg\neg C}{\underline{C}})}\,(3)}{\forall xB\to\neg\neg\forall xB(\frac{\neg\neg C}{\underline{C}})}\,(1)$$

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12. Lemma:

Let κ be a clear bar of S_B^{++}, then $\vdash \neg\neg B \to B(\frac{\kappa}{\neg\neg\kappa})$.

Proof:

Like the proof of prop. 7.

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13. Proposition:

If S_B^{++} has a clear bar free of x, then $\vdash_I \forall x \neg\neg B \to \neg\neg \forall xB$.

Proof:

By (12) and (3a) $\vdash_I \forall x \neg\neg B \to \forall xB(\frac{\kappa}{\neg\neg\kappa})$, where $\kappa = \langle \underline{C}_1, \ldots, \underline{C}_K \rangle$ is a clear bar of S_B^{++} free of x. K applications of (11) and (4f) yield the result.

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14. Corollary:

If for a formula A $\underline{\forall xB} \in S_A^{+} \Rightarrow S_B^{++}$ has a clear bar free of x, then $\vdash_C A \Rightarrow \vdash_I \neg\neg A$.

Proof:

By (10) and (13).

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15. Corollary (Cellucci 69):

If for every $\underline{\forall x B} \in S_A^+$ either $B \equiv \neg C$ or $Bx \equiv Cx \to D$ (D is free of x), then $\vdash_C A \Rightarrow \vdash_I \neg\neg A$.

Proof:

Use (14). In the first case $\langle \Lambda \rangle$ is a clear bar free of x for S_B^{++}, in the second - $\langle D \rangle$.

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16. Definitions:

A positive-chain in S_A is a sequence of consecutive elements $S_0 \leq \dots \leq S_K$ of S_A^+, and s.t. S_K is an end-point of S_A.

By the convention we have made to identify a $p \in S_A$ with its main logical symbol, if $\langle S_0, \dots, S_K \rangle$ is a possitive-chain, then $S_0 \dots S_{K-1}$ are logical symbols, and S_K is either Λ or a predicate letter.

If we assume that every $\underline{\neg B} \in S_A$ is writen as $\underline{B \to \Lambda}$, (as we do for the sequel), then no S_i $(1 \leq i \leq K)$ is a $\neg$-symbol.

Define now classes π_n $(0 \leq n)$ and σ_n $(1 \leq n)$ of positive-chains inductively:

(1) $\langle \Lambda \rangle \in \pi_0$

(2) $\langle P \rangle \in \sigma_1$

(3) $\langle t_1, \dots, t_m \rangle \in \{\begin{smallmatrix}\pi_n\\ \sigma_n\end{smallmatrix} \Rightarrow \langle \&, t_1, \dots, t_m \rangle \in \{\begin{smallmatrix}\pi_n\\ \sigma_n\end{smallmatrix}$ and $\langle \to, t_1, \dots, t_m \rangle \in \{\begin{smallmatrix}\pi_n\\ \sigma_n\end{smallmatrix}$

(4) $\langle t_1, \dots, t_m \rangle \in \sigma_n \Rightarrow \langle V, t_1, \dots, t_m \rangle \in \sigma_n$ and $\langle \exists x, t_1, \dots, t_m \rangle \in \sigma_n$

(5) $\langle t_1, \dots, t_m \rangle \in \pi_n \Rightarrow \langle \forall x, t_1, \dots, t_m \rangle \in \pi_n$

(6) $\langle t_1, \dots, t_m \rangle \in \pi_n \Rightarrow \langle V, t_1, \dots, t_m \rangle \in \sigma_{n+1}$ and $\langle \exists x, t_1, \dots, t_m \rangle \in \sigma_{n+1}$

(7) $\langle t_1, \dots, t_m \rangle \in \sigma_n \Rightarrow$
- $\langle \forall x, t_1, \dots, t_m \rangle \in \pi_n$ if some t_i $(1 \leq i \leq m)$ is a d-formula in which x is free
- $\langle \forall x, t_1, \dots, t_m \rangle \in \sigma_n$ otherwise.

We define classes η_n of formulas by

$$A \in \eta_m \equiv_{Df} \; m = \max\{n \mid <S_0 \ldots S_K> \text{ is a positive-chain in } S_A, \text{ and } <S_0 \ldots S_K> \in \{{}^{\pi}_{\sigma}{}^{n}_{n} \; .$$

17. Proposition:

If $A \in \eta_m$ and $\vdash_C A$, then m is a bound on the number of nested applications of the rule of double-negation (the Λ_C-rule of [Prawitz 65]) along any path in a classical proof of A in the natural-deduction system of [Prawitz 65].

Proof:

Let A be s.t. $\vdash_C A$, let $\{\underline{B}_1, \ldots, \underline{B}_K\} \subseteq S_A$ be the complete list of elements of S_A s.t. $\underline{\forall x_i B_i} \in S_A^+$, $\underline{B}_0 \equiv_{Df} \underline{A}$, $\beta \equiv_{Df} <\underline{B}_0, \underline{B}_1, \ldots, \underline{B}_K>$ and $\overline{A} = A\left(\frac{\beta}{\neg\neg\beta}\right)$.

By (15) $\vdash_I \overline{A}$.

Let $T_i =_{Df} S^{++}_{\underline{B}_i}$ and κ_i be the maximal clear bar of T_i $(0 \leq i \leq K)$.

$\kappa =_{Df} \bigcup_{i=0}^{K} \kappa_i$ (set-theoretic union).

By (11), (12) and (3a) $\vdash_I \overline{A} \Rightarrow \vdash_I \hat{A}$, where $\hat{A} \equiv A\left(\frac{\kappa}{\neg\neg\kappa}\right)$.

Let γ be a maximal positive chain in S_A, $\gamma = <t_1 \ldots t_m> \in \{{}^{\pi}_{\sigma}{}^{n}_{n}$.

Call a subchain $<t_j, \ldots, t_k>$ $(1 \leq j < k \leq m)$ of γ a d-block if:

(i) for some $j \leq i \leq k$ t_i is a d-formula

(ii) for no $j \leq i \leq k$ t_i is an "effective" universal-formula, i.e. - a $\forall x$-formula s.t. x occurs free in some t_l $(i < l \leq m)$ which is a d-formula.

(iii) $<t_j \ldots t_k>$ is maximal in γ with repsect to properties (i) and (ii).

A routine induction on (16) and the construction of $\hat{A}$ above yields:

n = the number of d-blocks in γ

= the number of double-negations along γ in $\hat{A}$.

To prove now the proposition, begin a deduction with $\hat{A}$, and split it, using the elimination rules. Whenever a $\underline{\neg\neg D} \in \neg\neg K$ appears, use the rule of double-negation to replace it by $\underline{D}$. When all the elements of κ are treated, reconstruct A.

For any positive chain γ, its initial segment ending with the first element of the last d-block in it (= the last element of $K \cap \gamma$) is a segment of the E-part of some path δ in the deduction $\begin{matrix}\hat{A}\\(\Pi)\\A\end{matrix}$ described above; thus the number of applications of the rule of double-negation along δ = the number of d-block in δ = the index of the σ_n (or π_n) class to which it belongs. This concludes the proof, since $\vdash_I \hat{A}$, and therefore we have a deduction Σ without applications of the rule of double-negation s.t. $\begin{matrix}\Sigma\\\hat{A}\\\Pi\\A\end{matrix}$ is a proof.

$\boxtimes$

18. We cannot expect to have a complete structural description which will give for every $A \in C$ a set $\kappa \subseteq S_A$ s.t. $\vdash_C A \Rightarrow \vdash_I A\,(\frac{\kappa}{\neg\neg\kappa})$, and which is minimal in that respect, i.e. - for every $\beta \subsetneq K$ $\nvdash_I A\,(\frac{\beta}{\neg\neg\beta})$.

Such a description would yield immediately a decision for I:
Given A, take $D^A \equiv_{Df} A \vee \neg A$.
We can, by our assumption, find effectively a $\kappa \subseteq S_{D^A}$ s.t. $\vdash_I D^A(\frac{\kappa}{\neg\neg\kappa})$ but for every $\beta \subsetneq K$ $\nvdash_I D^A(\frac{\beta}{\neg\neg\beta})$.
Now, if $\kappa = \emptyset$, then $\vdash_I D^A$, hence $\vdash_I A$ or $\vdash_I \neg A$, and it can be decided effectively which case holds.
If $\kappa \neq \emptyset$, then $\nvdash_I A$, for otherwise $\vdash_I D^A$, construdicting the minimality of κ.

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